

Points at distance 1

Five problems for curious students and their teachers

In 1946 Paul Erdős asked: if you place n points in the plane, how many pairs of them can be exactly 1 apart? Write $u(n)$ for the largest possible number. Square grids give about $n^{1+c/\log\log n}$ pairs, and Erdős conjectured that one cannot do much better. In May 2026 OpenAI announced that one of its AI models had disproved this conjecture, with arrangements that have at least $n^{1+\delta}$ such pairs for a fixed $\delta > 0$; mathematicians have checked the proof. The true size of $u(n)$ is still unknown. The first problem is the question from the video.

1 Four points

WARM-UP

What is the largest number of pairs at distance exactly 1 among four points in the plane? Find an arrangement and explain why you cannot do better.

2 Build them up

WARM-UP

- Find six points with nine pairs at distance 1.
- Find seven points with twelve pairs at distance 1.
- Show that for every $n \geq 2$ there are n points with at least $2n - 3$ pairs at distance 1.

3 Erdős's grid trick

MEDIUM

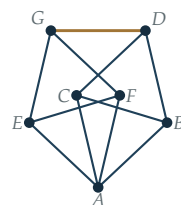
- How many points with integer coordinates lie at distance 5 from the origin?
- In a 13×13 square grid (169 points), count the pairs at distance exactly 1 and the pairs at distance exactly 5. Shrinking the grid by a factor of 5 turns the second number into unit distances.
- Find a distance with 16 grid points around the origin.

4 The Moser spindle

MEDIUM

Take two copies of the rhombus from Problem 1, made of two equilateral triangles, with a common vertex A . Rotate one copy about A until the far tips D and G are exactly 1 apart.

- How far is each tip from A ? Find the angle of rotation.
- Count the pairs at distance 1 among these seven points.
- Show that the seven points cannot be colored with three colors so that every two points at distance 1 get different colors.



5 An upper bound

CHALLENGE

Fix n points in the plane, let u be the number of pairs at distance 1, and let $d(p)$ be the number of points at distance 1 from the point p .

- Explain why two different points have at most two common points at distance 1 from both.
- Count the triples $(p, \{q, r\})$, where q and r are both at distance 1 from p , in two ways, and show that $\sum_p d(p)(d(p) - 1) / 2 \leq n(n - 1)$.
- Use $\sum_p d(p) = 2u$ and $\sum_p d(p)^2 \geq (\sum_p d(p))^2 / n$ to show that $u < n^{3/2}$.

Hints and answers

Points at distance 1



1 Four points

HINT Could all six pairs be at distance 1? Start with three points at mutual distance 1.

Five: two equilateral triangles with side 1 sharing a side form a rhombus whose four sides and short diagonal have length 1. Six is impossible: if A, B, C form an equilateral triangle with side 1, the only other point at distance 1 from both A and B is the mirror image D of C in the line AB , and $|CD| = \sqrt{3}$.

2 Build them up

HINT Use equilateral triangles with side 1. For (b), think of a regular hexagon.

- (a) A triangle with side 2 cut out of the triangular grid: its three corners and three midpoints give nine unit segments.
- (b) A regular hexagon with side 1 together with its center: six sides and six spokes.
- (c) A strip of equilateral triangles: each new point forms an equilateral triangle with the previous two and adds two unit distances, so $1 + 2(n - 2) = 2n - 3$.

3 Erdős's grid trick

HINT (a) $25 = 5^2 + 0^2 = 3^2 + 4^2$. (b) A vector (a, b) fits into a $k \times k$ grid in $(k - |a|)(k - |b|)$ positions. (c) Look for a number that is a sum of two squares in two different ways.

- (a) 12: $(\pm 5, 0), (0, \pm 5), (\pm 3, \pm 4), (\pm 4, \pm 3)$.
- (b) Distance 1: $2 \cdot 13 \cdot 12 = 312$ pairs. Distance 5 (six vectors up to sign): $2 \cdot 13 \cdot 8 + 4 \cdot 10 \cdot 9 = 568$ pairs.
- (c) For example $\sqrt{65}$, because $65 = 1^2 + 8^2 = 4^2 + 7^2$. Erdős used distances with more and more such representations, which gives about $n^{1+c/\log \log n}$ unit distances.

4 The Moser spindle

HINT (a) The long diagonal of the rhombus. (c) In one rhombus, the two middle points get two different colors. What does that force for A and D ?

- (a) The tips are $\sqrt{3}$ from A , so the angle θ satisfies $2\sqrt{3} \sin(\theta/2) = 1$, which gives $\theta \approx 33.56^\circ$.
- (b) $5 + 5 + 1 = 11$.
- (c) B and C have different colors, and A and D are both at distance 1 from B and C , so both get the third color. Likewise A and G share a color, so D and G do too, although $|DG| = 1$. Found by Leo and William Moser in 1961, this shows that the plane needs at least four colors; in 2018 Aubrey de Grey showed that four are not enough.

5 An upper bound

HINT (a) Two circles of radius 1. (b) Each pair $\{q, r\}$ is counted at most twice. (c) You should get $2u^2/n - u < n^2$; solve this quadratic inequality for u .

- (a) The points at distance 1 from q and from r lie on two different circles of radius 1, which meet in at most two points.
- (b) For a fixed p there are $d(p)(d(p) - 1)/2$ pairs $\{q, r\}$; for a fixed pair $\{q, r\}$ there are at most two points p ; and there are $n(n - 1)/2$ pairs.
- (c) From (b), $(\sum d(p)^2 - \sum d(p))/2 \leq n(n - 1) < n^2$, so $4u^2/n - 2u < 2n^2$, that is $2u^2 - nu - n^3 < 0$. Hence $u < (n + \sqrt{n^2 + 8n^3})/4 \leq n^{3/2}$. Erdős proved this kind of bound in 1946; the best known today, of order $n^{4/3}$, is from 1984.

Further reading: [OpenAI's announcement of the 2026 result](#) (20 May 2026).