

Electrons on a sphere

Five problems for curious students and their teachers

Put N electrons on a sphere of radius 1. Every pair repels, and the energy of an arrangement, $E = \sum_{i < j} 1/|x_i - x_j|$, is the sum of $1/d$ over all pairs, where d is the distance between the two electrons. The Thomson problem asks for the arrangement with the smallest energy. School mathematics is enough for everything below; a calculator helps in Problem 3.

1 Two electrons

WARM-UP

Two electrons sit on the sphere. Their energy is $1/d$. Where should they sit to make the energy as small as possible, and what is the smallest energy?

2 Count the pairs

WARM-UP

- How many pairs are there among N electrons? How many among 400?
- Six electrons sit at the vertices of a regular octahedron, $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$. Show that their energy is $6\sqrt{2} + \frac{3}{2} \approx 9.9853$.
- Five electrons: one at each pole and three forming an equilateral triangle on the equator. Show that their energy is $\frac{1}{2} + 3\sqrt{2} + \sqrt{3} \approx 6.4747$.

3 The cube is not the answer

MEDIUM

Eight electrons sit at the vertices of a cube inscribed in the sphere, so the edge of the cube is $2/\sqrt{3}$.

- Show that their energy is $6\sqrt{3} + 3\sqrt{6} + 2 \approx 19.7408$.
- Turn the top face of the cube by 45° about the vertical axis. Find the new distances and show that the energy drops to about 19.6816.

4 A lower bound that settles $N = 2, 3, 4$

CHALLENGE

Let $x_1, \dots, x_N$ be points on the unit sphere, let $M = N(N-1)/2$ be the number of pairs and let E be their energy.

- Show that $\sum_{i < j} |x_i - x_j|^2 = N^2 - |x_1 + \dots + x_N|^2$. In particular, this sum is at most N^2 .
- Using the inequalities between the harmonic, arithmetic and quadratic means of the M distances, show that $E \geq M^{3/2}/N$.
- Check that two opposite points, an equilateral triangle on a great circle and a regular tetrahedron reach this bound. Why does this prove that they are the best arrangements for $N = 2, 3, 4$?
- Explain why the bound is never reached when $N \geq 5$.

5 Euler's formula and the neighbors

MEDIUM

Join neighboring electrons by edges so that the sphere is divided into triangles, each edge shared by two triangles. Let V , E , F be the numbers of vertices (electrons), edges and triangles (here E is not the energy), and let $\deg v$ be the number of neighbors of v .

- Show that $3F = 2E$ and that the degrees add up to $2E$.
- Use Euler's formula $V - E + F = 2$ to show that $\sum_v (6 - \deg v) = 12$.
- Suppose every electron has 5, 6 or 7 neighbors. Show that there are exactly 12 more electrons with five neighbors than with seven.
- Show that at least four electrons always have fewer than six neighbors.

Bonus for programmers: scatter N points on the sphere, move each a small step along the total repulsive force, push it back onto the sphere, repeat. Try $N = 5, 7, 8$.

Hints and answers

Electrons on a sphere

1 Two electrons

HINT How far apart can two points of the sphere be?

At opposite ends of a diameter. Since $d \leq 2$, the energy is at least $\frac{1}{2}$, with equality exactly when $d = 2$.

2 Count the pairs

HINT Sort the pairs by distance: in (b) there are two different distances, in (c) three.

- (a) $N(N-1)/2$; for 400 electrons that is 79 800 pairs.
- (b) 12 pairs at distance $\sqrt{2}$ and 3 pairs of opposite vertices at distance 2: $12/\sqrt{2} + 3/2 = 6\sqrt{2} + \frac{3}{2}$.
- (c) The poles are 2 apart, each pole is $\sqrt{2}$ from each equatorial electron (6 pairs), and the triangle has side $\sqrt{3}$ (3 pairs): $\frac{1}{2} + 6/\sqrt{2} + 3/\sqrt{3} = \frac{1}{2} + 3\sqrt{2} + \sqrt{3}$.

3 The cube is not the answer

HINT The squares stay at heights $\pm 1/\sqrt{3}$, at distance $\sqrt{2/3}$ from the axis. Two points on different squares, whose positions differ by an angle θ around the axis, are at distance d with $d^2 = \frac{4}{3}(2 - \cos \theta)$.

- (a) There are 12 edges of length $2/\sqrt{3}$, 12 face diagonals of length $2\sqrt{2}/\sqrt{3}$ and 4 long diagonals of length 2, so $E = \frac{12\sqrt{3}}{2} + \frac{12\sqrt{3}}{2\sqrt{2}} + \frac{4}{2} = 6\sqrt{3} + 3\sqrt{6} + 2$.
- (b) Inside each square nothing changes (4 sides $2/\sqrt{3}$, 2 diagonals $2\sqrt{2}/\sqrt{3}$). Between the squares the angles are now 45° and 135° , eight times each, giving distances ≈ 1.3130 and ≈ 1.8999 . So $E \approx 9.3777 + 8/1.3130 + 8/1.8999 \approx 19.6816$. The twist removes the four short vertical edges. Moving the two squares slightly closer together lowers the energy further, to 19.6753: this is the square antiprism, proved optimal in 2026.

4 A lower bound that settles $N = 2, 3, 4$

HINT (a) Expand $|x_i - x_j|^2 = 2 - 2x_i \cdot x_j$ and compare with $|x_1 + \dots + x_N|^2$. (b) For distances $d_1, \dots, d_M$: $\sum 1/d_k \geq M^2 / \sum d_k$ and $\sum d_k \leq \sqrt{M \sum d_k^2}$.

- (a) $\sum_{i < j} (2 - 2x_i \cdot x_j) = N(N-1) - (|\sum x_i|^2 - N) = N^2 - |\sum x_i|^2$.
- (b) $E \geq M^2 / \sum d_k \geq M^2 / \sqrt{M \cdot N^2} = M^{3/2} / N$.
- (c) The bound equals $\frac{1}{2}$, $\sqrt{3}$ and $\frac{3\sqrt{6}}{2} \approx 3.6742$ for $N = 2, 3, 4$, exactly the energies of these three arrangements. No arrangement can go below the bound, so these are optimal.
- (d) Equality forces all M distances to be equal, and at most four points in space are pairwise equidistant. For five electrons the bound is 6.3246, while the best energy is 6.4747.

5 Euler's formula and the neighbors

HINT (b) Write $6V - 2E$ using (a) and Euler's formula. (d) Every electron has at least three neighbors.

- (a) Every triangle has three edges and every edge lies in two triangles; every edge has two ends.
- (b) $F = 2E/3$, so $V - E/3 = 2$, that is $6V - 2E = 12$; and $\sum_v (6 - \deg v) = 6V - 2E$.
- (c) Electrons with six neighbors contribute 0, with five +1, with seven -1.
- (d) The positive terms add up to at least 12 and each is at most 3, so at least four electrons have fewer than six neighbors. The tetrahedron shows that four can happen. This is the same count that forces twelve pentagons on a soccer ball.

The full story, including the computer-checked proofs of 2026 for seven, eight and nine electrons:

maciejstarostka.com/thomson-problem.html